Computing the Radius of the Earth

What did Eratosthenes do?
There is another way of doing it.

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Prologue. Eratosthenes

More than 2200 years ago, the Greek mathematician Eratosthenes computed the circumference U of the Earth (and thus the radius R) by an ingenious method, and probably was the first one to achieve this. He was the head of the famous library of Alexandria in Egypt. In the South of Egypt, the city of Syene (now Assuan) was situated on the banks of the river Nile. Eratosthenes took Alexandria and Syene as reference points for his measurement. He assumed that both cities lay on the same meridian and that Syene lay on the Tropic of Cancer (the northern tropic). Furthermore, Eratosthenes knew the distance between the two cities. In Figure 0.1. we find a sketch of Eratosthenes' mathematical model: A cross section through the Earth along the Alexandria-Syene meridian with the Sun standing perpendicularly above Syene and standing at an angle α from the zenith in Alexandria. We see the sunrays shining onto the two cities when the Sun reaches his highest position at the begin of summer. As the sunrays arrive in parallels we also find α as central angle which can be taken as a measure for the distance between the cities. Eratosthenes found $\alpha = 7.2°$ which gave this distance as 1/50 of the circumference of the Earth. And this result is astonishingly good! Eratosthenes measured the distance Alexandria - Syene and the circumference in *stadia* and we do not know exactly which length a *stadion* in ancient Egypt had. In modern length units one would get a circumference of approx. 42000 km.

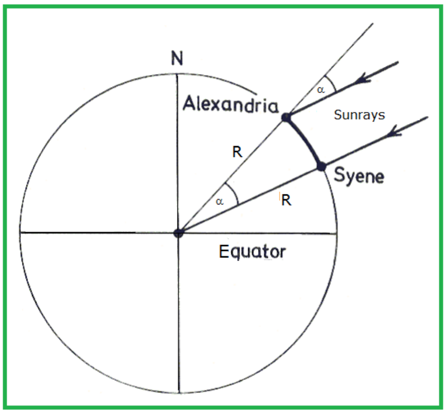


Figure 0.1. Eratosthenes' mathematical model for measuring the circumference of the Earth (not to scale)

Eratosthenes lived in the 3rd century B.C. Since long before his lifetime, Greek philosophers assumed that the Earth was spherical. It's a myth that in hellenistic times and even much later in the Middle Ages the notion of a flat Earth was common belief.

Quite naturally, Eratosthenes' measurement and calculation could not be totally accurate. His assumptions - the Earth being a perfect sphere and the parallelity of the sunrays - are not exactly correct. Furthermore, Syene does not lie on the same meridian as Alexandria and does not lie on the Tropic of Cancer (but nearly does).

1. A Different Idea

Eratosthenes' mathematical reasoning and the measurements he took can be quite easily reproduced nowadays. But are there other ways to compute the radius of the Earth? Well, there certainly are, and one very interesting and quite surprising idea appeared in Colin Wright's blog [1]. This blog gives a presentation of the idea in several stages (you can study them backwards beginning with [2]). In his blog [3], Colin Beveridge contributed to the discussion of the idea [4]. - Both blogs belong to my favourites and are recommended to the attention of the reader.

Now what is this idea? Is it necessary to take measurements at two different places? Or is there a way to do it locally? The way of reasoning starts with the conjecture that at sunrise on the beach, the time needed for the sunrays wandering down a pole depends on the radius of the Earth. This conjecture includes knowledge about the date and the observer's latitude.

This idea proved to be fruitful and practicable. Chapters 4. and 5. will show how it works.

2. Geocentric Coordinates

The following chapters use a geocentric model [5]. This model uses spherical trigonometry and allows to describe the relative positions of Earth and Sun in a straightforward way. This description contains *equatorial* [6] and *horizontal* [7] coordinates; see chapter 5. The geocentric model is geometrically equivalent with other astronomical models, in particular with the heliocentric model with the Sun in the center - both models give valid descriptions of the relative movements of Earth and Sun.

The figures in chapters 4. and 5. which show the Earth as well as the Sun are based on the geocentric model. But these figures and the related computations contain some simplifying assumptions: The center of the celestial sphere is the Earth which is thought as a mere point. The Sun moves on the celestial sphere and is also thought as reduced to a point. At equinox the Sun describes a great circle, and apart from equinox a small circle whose plane is parallel to the plane of the great circle. - In this simplified model, the deviation from reality is negligible as far as our goal, the measuring of the radius of the Earth, is concerned. The small deviation is - among other reasons - due to the fact that the celestial sphere is not exactly spherical because the distance Earth - Sun is not constant, and of course Earth and Sun are not points. But the radii of these bodies are very small compared to their distance.

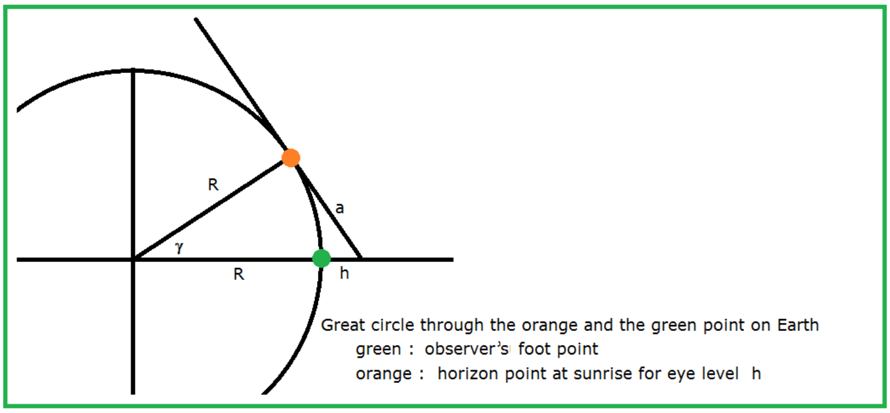
3. Distance of the horizon

Figure 3.1. describes the moment when someone who is at a sea shore sees the Sun rising over the horizon. Her or his eyes are at height h. The upper sketch shows the radius R of the Earth, the distance of the horizon a and the corresponding angle γ. The lower sketch shows that the Sun is standing at angle γ below the tangential plane at the observer's foot.

(3.1) $a = \sqrt{2Rh + h^2} \approx \sqrt{2Rh}$ (approximation for small h)

$$\text{(3.2)} \quad \sin\gamma = \frac{a}{R+h} \qquad \gamma \approx \sin\gamma = \frac{\sqrt{2Rh+h^2}}{R+h} \quad \text{(approximation for small } \gamma\text{)}$$

$$\text{(3.3)} \quad \gamma \approx \sqrt{\frac{2h}{R}} \quad \text{in radians} \quad \text{(approximation for small } \gamma,\ h\text{)}$$



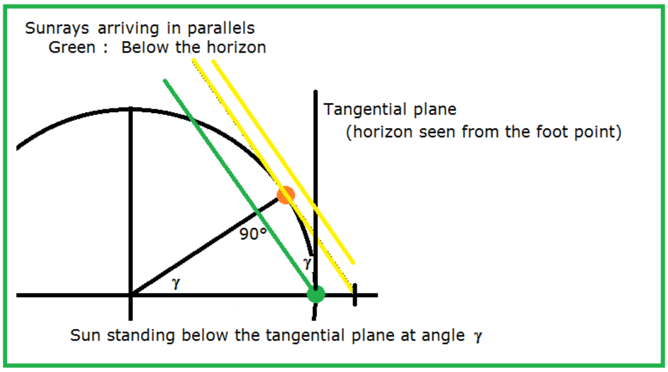


Figure 3.1. Horizon

In the strict sense, γ should be negative because the Sun stands below the horizon. In this chapter, the sign is of no importance. We shall change the sign in chapter 5.

4. Radius of the Earth calculated at equinox

Figure 4.1. shows, somewhat simplified, the perspective of an observer A at sea level. He sees the Sun rising at the intersection of his horizon with the Sun's path along the celestial sphere. The corresponding angle $90° - \phi$ only depends on A's latitude ϕ, and not on the time of the year. The orange lines refer to the position of the Sun at the moment when an observer B sees the Sun rise at the same place on the beach but from height h. At this moment, for A the Sun ist still below the horizon; the Sun will move along the path β, thus connecting the two risings for B and A.

In figure 4.1. β and γ are parts of curves on the celestial sphere. They require the application of spherical trigonometry; parts of great circles are given as angles. Figure 4.2. shows that β is not part of a great circle, except at equinox when the path of the Sun (green-orange) goes along the equator of the celestial sphere (declination $\delta = 0$). For this reason, this chapter only deals with the equinox. (Figure 4.2. shows the general case; the path of the Sun goes along a more northernly small circle; in chapter 5., we will come back to this case.)

Spherical trigonometry provides $cos\ \phi = tan\ \gamma / \tan\ \beta \approx \gamma/\beta$ for small angles. This approximation matches with the corresponding result in plane trigonometry. Now everything is prepared for the calculation of R.

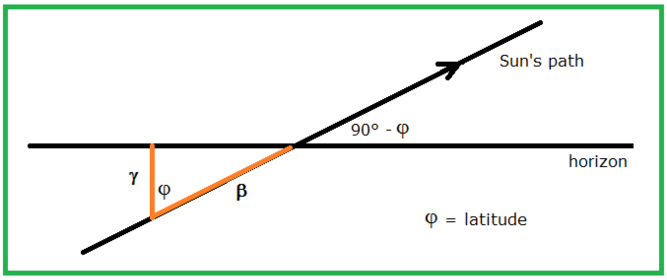


Figure 4.1. Sunrise

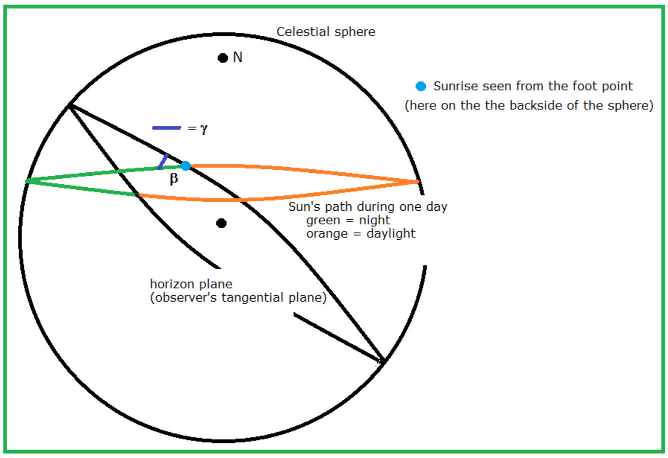


Figure 4.2. Celestial sphere

(4.1) $\beta = \dfrac{\gamma}{cos\ \phi} \ \approx \ \dfrac{\sqrt{2h/R}}{cos\ \phi}$ (figure 4.1. und (3.3))

(4.2) $t = \dfrac{\beta}{2\pi} \ \approx \ \dfrac{\sqrt{h/2R}}{\pi\ cos\ \phi}$

(4.3) $R \ \approx \ \dfrac{h}{2\pi^2\ cos^2\phi\ \ t^2}$ $U \ \approx \ \dfrac{h}{\pi\ cos^2\phi\ \ t^2}$

(4.4) $R \ \approx \ \dfrac{(\sqrt{h_1} - \sqrt{h_2})^2}{2\pi^2\ cos^2\phi\ \ (t_1 - t_2)^2}$ $U \ \approx \ \dfrac{(\sqrt{h_1} - \sqrt{h_2})^2}{\pi\ cos^2\phi\ \ (t_1 - t_2)^2}$

Colin Wright got the same results by pursuing another reasoning (see [2]).

In (4.2) t is the time measured in days for the sunrays going down a pole of height h. Remember that $t = \beta/(2\pi)$ only holds at equinox because 2π stands for the full circumference of a great circle.

In (4.4) time measurements t_1 and t_2 are taken for sunrise watched from heights h_1 and h_2.

(4.3) and (4.4) provide a first result how we can estimate the radius of the Earth without having to move very much. The calculation requires the knowledge of our latitude (not a great problem - just look at the North Star); and the measurement must be taken at equinox.

5. General calculation of the radius of the Earth

For $\delta \neq 0$, figure 4.1. can no longer be applied. Figure 5.1. shows the reason: Now, the path of the Sun in figure 4.1. ist not a great circle any more; instead, it goes along the small circle shown in figures 4.2. and 5.1. The blue arrow in figure 5.1. points to the critical part of the path. If we would take figure 4.1. as a basis for further calculation this would amount to describing the path of the Sun as a great circle with identical rising angle $90° - \phi$. This great circle (which we cannot use) is also included in figure 5.1. - So we need another approach than in chapter 4. We will use equatorial and horizontal coordinates.

The **equatorial system** describes the Sun's position by *declination* $\delta \in [-23,43°, +23,43°]$ and the *time of the day* $\tau \in [\ 0°, 360°)$; figure 5.1. is an example for this system.

Cartesian coordinates in the equatorial system:

$\tilde{x} = -cos\ \delta\ cos\ \tau$
$\tilde{y} = cos\ \delta\ sin\ \tau$
$\tilde{z} = sin\ \delta$

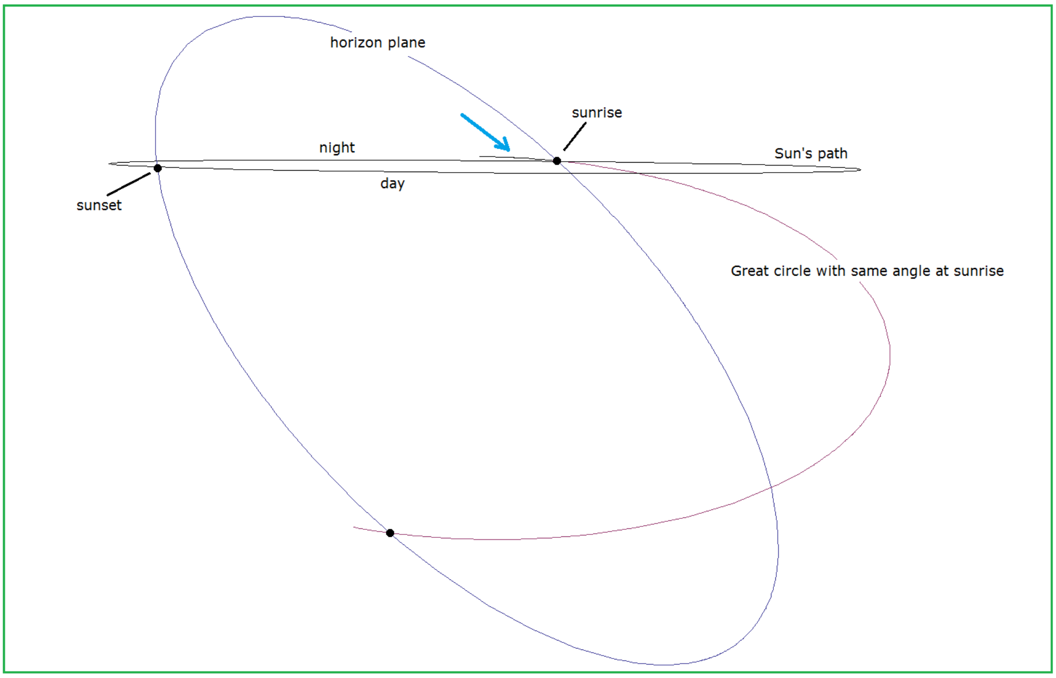


Figure 5.1. Horizon plane and the path of the Sun

In the **horizontal system** the horizon plane lies horizontally, and compared to the equatorial system, tilted by $90° - \phi$. The position of the Sun is described by his *height* $\gamma \in [\,-90°, +90°]$ above or under the horizontal plane and his *geographic compass direction* $A \in [\,0°, 360°)$ as viewed by an observer on Earth.

Cartesian coordinates in the horizontal system:

$x = -cos\ \gamma\ cos\ A$
$y = cos\ \gamma\ sin\ A$
$z = sin\ \gamma$

Conversion of the cartesian coordinates:

$x = \tilde{x}\ sin\ \phi - \tilde{z}\ cos\ \phi$
$y = \tilde{y}$
$z = \tilde{x}\ cos\ \phi + \tilde{z}\ sin\ \phi$

This gives:

(5.1) $sin\ \gamma\ =\ z\ =\ \tilde{x}\ cos\ \phi\ +\ \tilde{z}\ sin\ \phi\ =\ -cos\ \delta\ cos\ \tau\ cos\ \phi\ +\ sin\ \delta\ sin\ \phi$

$$(5.2)\quad \tau \;=\; arccos\,\frac{sin\;\delta\;sin\;\phi\;-\;sin\;\gamma}{cos\;\delta\;cos\;\phi}\;\approx\; arccos\,\frac{sin\;\delta\;sin\;\phi\;+\;\sqrt{2h/R}}{cos\;\delta\;cos\;\phi}$$

(approximation according to (3.2), (3.3))

γ in (5.2) has a different sign as in chapter 3. because the Sun stands below the horizon.

$$(5.3)\quad \delta = 0 \quad\longrightarrow\quad \tau \;\approx\; arccos\,\frac{\sqrt{2h/R}}{cos\;\phi}$$

$$(5.4)\quad R \;\approx\; \frac{2h}{(cos\;\tau\;cos\;\delta\;cos\;\phi\;-\;sin\;\delta\;sin\;\phi)^2}$$

$$(5.5)\quad \delta = 0 \quad\longrightarrow\quad R \;\approx\; \frac{2h}{(cos\;\tau\;cos\;\phi)^2}$$

τ in (5.2) - (5.5) is not a time interval but an instant of time given in $[0, 2\pi)$ (one day). For obtaining t as in chapter 4. (unit: one day) one has to subtract the right side of (5.2) from the τ-value for $h = 0$ and then to divide by $[0, 2\pi)$:

$$(5.6)\quad t \;\approx\; \frac{1}{2\pi}\,(arccos(tan\;\delta\;tan\;\phi) - arccos\,\frac{sin\;\delta\;sin\;\phi\;+\;\sqrt{2h/R}}{cos\;\delta\;cos\;\phi}\;)$$

$$\Longrightarrow\quad \sqrt{2h/R} \;\approx\; cos(arccos(tan\;\delta\;tan\;\phi) - 2\pi t)\;cos\;\delta\;cos\;\phi - sin\;\delta\;sin\;\phi$$

$$(5.7)\quad \delta = 0 \quad\longrightarrow\quad t \;\approx\; \frac{1}{2\pi}\;arcsin\,\frac{\sqrt{2h/R}}{cos\;\phi}$$

$$(5.8)\quad R \;\approx\; \frac{2h}{(cos(arccos(tan\;\delta\;tan\;\phi)\;-\;2\pi t)\;cos\;\delta\;cos\;\phi\;-\;sin\;\delta\;sin\;\phi)^2}$$

$$(5.9)\quad \delta = 0 \quad\longrightarrow\quad R \;\approx\; \frac{2h}{(sin\;2\pi t\;cos\;\phi)^2}$$

(4.2) is an approximation for (5.7) using $arcsin\;x \;\approx\; x$ for small x.
(4.3) is an approximation for (5.9) using $sin\;x \;\approx\; x$ for small x.

Application

All calculations should be performed only for small heights h in order for the approximative formulae to be sufficiently exact.

Equinox :

With stopwatch : (4.4) is the most suitable formula for the calculation of R. If the second time measurement is taken at the foot of the pole ($h_2 = 0$) take (4.3) or (5.9).

With clock : If a clock showing local time is available **a single measurement will suffice!** (5.5) provides the result.

Off-equinox seasons :

With stopwatch : (5.8) should be applied. The second measurement has to be taken at the foot of the pole. - For two heights $h_1 > h_2 > 0$ as in (4.4) we get by (5.6)

$$(5.10) \quad t_1 - t_2 \approx \frac{1}{2\pi} \left(arccos \frac{sin\ \delta\ sin\ \phi\ +\ \sqrt{2h_2/R}}{cos\ \delta\ cos\ \phi} - arccos \frac{sin\ \delta\ sin\ \phi\ +\ \sqrt{2h_1/R}}{cos\ \delta\ cos\ \phi} \right)$$

and can numerically compute R.

With clock : If a clock showing local time is available **a single measurement will suffice!** (5.4) provides the result.

Example 1

$\phi = 54.19°$ (Heligoland, East coast) $\delta = -5°$ (8th March)

We take the time between the Sun's touching the top resp. the foot of a pole at the shore :

$h = 6.1\ m \qquad t = 33\ sec \qquad \longrightarrow \qquad R \approx 6326.44\ km$ (by (5.8))

Example 2

$\phi = 47.67°$ (Lake Constance, South-East bank of Constance town) $\delta = -20°$ (22nd November)

We take the local time when the Sun touches the top of a flagpole at the shore :

$h = 14\ m \qquad \tau = 113.35° = 7:33:24$ am $\longrightarrow \qquad R \approx 6699.51\ km$ (by (5.4))

Example 3

$\phi = 37.06°$ (Syracuse, Sicily, East coast) $\delta = 0°$ (20th March)

We take the time between the Sun's touching two different heights of a house at the shore :

$h_1 = 11\ m \quad h_2 = 5\ m \qquad t_1 - t_2 = 10.5\ sec \qquad \longrightarrow \qquad R \approx 6289.31\ km$ (by (4.4))

Example 4

$\phi = 32.77°$ (Porto da Cruz, Madeira, North-East coast) $\delta = 21.3°$ (17th July)

We take the time between the Sun's touching two different heights of a pole at the shore :

$h_1 = 10\,m \quad h_2 = 2.5\,m \qquad t_1 - t_2 = 16\ sec \qquad \longrightarrow \qquad R \approx 6427.85\,km$ (numerically by (5.10))

How precise must the measurements be?

When taking actual measurements, one will face the problem that the shadow line is not sharp. The taking of the time will thus be quite inaccurate. The precision of the result (the radius of the Earth) can be estimated by the examples above. Let's do it for example 1: A deviation of 1 *sec*, amounting to approx. 3% of the time measurement, results in an error of approx. 6% for the radius.

References

[1] `https://www.solipsys.co.uk/new/ColinsBlog.html`

[2] `https://www.solipsys.co.uk/new/EarthRadiusRefined.html?ColinsBlog`

[3] `https://www.flyingcoloursmaths.co.uk/blog/`

[4] `http://www.flyingcoloursmaths.co.uk/stab-colins-puzzle`

[5] `https://en.wikipedia.org/wiki/Celestialcoordinatesystem`

[6] `https://en.wikipedia.org/wiki/Equatorialcoordinatesystem`

[7] `https://en.wikipedia.org/wiki/Horizontalcoordinatesystem`

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